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NONLINEAR FLEXURAL VIBRATIONS OF LAYERED PLATES†

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Abstract—By means of a higher-order plate-bending theory developed by Whitney and Pagano along with Berger's hypothesis, large amplitude forced vibrations of moderately thick laminated specially orthotropic plates are investigated. The theory includes shear deformation and rotatory inertia in the same manner as Mindlin's theory for isotropic homogeneous plates. The in-plane forces due to large deflections are assumed to be constant within the plate domain. Considering time-harmonic forcing a Kantorovich–Galerkin procedure provides the formulation of this problem in the lower band of the frequency domain. In the case of laminates made of isotropic layers an analogy to thin homogeneous plates is given, which is complete in the case of polygonal planforms and hard hinged supports. Furthermore, the plate deflection is determined by the solution of two (second-order) Helmholtz–Klein–Gordon boundary value problems. Inserting these results into a proper domain integral leads to Berger's normal force. This problem-oriented strategy renders the nonlinear frequency response functions of deflection of the undamped layered plate.

1. INTRODUCTION

Dynamic problems of homogeneous moderately thick plates are efficiently dealt with according to the theory of Mindlin (1951), which yields precise approximations in comparison with solutions deduced from the three-dimensional equations of elasticity within a wide range of the thickness–span ratio. Herrmann and Armenakas (1960) established a linear vibration theory of shear deformable plates under initial stress. An extension of Mindlin's theory to heterogeneous anisotropic plates has been derived by Whitney and Pagano (1970), which will be used in the following to determine flexural vibrations of polygonal layered plates with hard hinged supports.

Taking into account large flexural vibrations we make use of Berger's approximation: Berger (1955) derived a decomposition of v. Karman's system of two coupled nonlinear differential equations for plate deflection and stress function by neglecting the strain energy density due to the second invariant of the plate's middle-surface strain. Since the resulting equation is uncoupled and quasi-linear it has been extensively applied in place of the complete set of v. Karman's equations; the influence of the v. Karman's in-plane forces is replaced by tensile forces of hydrostatic type being constant within the whole plate domain. Berger's equation has frequently been discussed for static and dynamic problems of homogeneous plates, see e.g. Nowinski and Ohnabe (1972), Schmidt (1974), Wah (1963), Wu and Vinson (1969a), Irschik (1990), and Heuer *et al.* (1990), and it has also been applied to vibration problems of laminates by Wu and Vinson (1969b, 1970) and Bennett (1971). The dynamic analog of the v. Karman theory for homogeneous thin plates has been derived by Herrmann (1955) and extended to laminated anisotropic plates by Whitney and Leissa (1969). It is now widely accepted that Berger's hypothesis gives a fairly good approximation to the problem, provided that the in-plane displacements are restrained on the boundary.

Subsequently, the nonlinear coupled equations of motion are summarized for symmetric laminates composed of multiple specially orthotropic layers. A harmonic uniform force-excitation is assumed and application of the Kantorovich–Galerkin procedure (see Szemplinska-Stupnicka, 1983) yields the boundary value problem which is to be solved for the undamped frequency response functions.

In the case of plates made of multiple isotropic layers the boundary value problem reduces to that of a homogeneous Mindlin-plate with effective stiffnesses. After eliminating

† Dedicated to Professor George Herrmann on the occasion of his 70th birthday.

the cross-sectional rotations a single fourth-order equation for the deflection is obtained. As shown in earlier works of Irschik (1985) and Irschik et al. (1989), this problem can be decomposed into two Helmholtz equations with homogeneous boundary conditions, which finally are solved iteratively along with the computation of Berger's deflection-dependent in-plane force.

In the numerical example the nonlinear frequency response functions of a composite plate strip are worked out in detail.

2. EQUATIONS OF MOTION FOR NONLINEAR VIBRATIONS OF MODERATELY THICK AND LAYERED PLATES

We consider a laminated plate with N layers of specialized orthotropy where the elastic properties are distributed symmetrically with respect to the middle surface. According to the theory of Whitney and Pagano (1970) for shear deformable layered plates with rotatory inertia, which is based on linear distribution of the in-plane displacements in the thickness direction, plate motions may be expressed by a sixth-order system of differential equations in terms of the deflection of the plate's middle plane $\bar{w}$ and the cross-sectional rotations $\bar{\psi}_x$, $\bar{\psi}_y$. Also taking into account hydrostatic inplane forces $\bar{\bar{n}}$ renders (compare Kollbrunner and Herrmann (1956), and Irschik (1985) for homogeneous plates):

$$\mathscr{S}_1\{\bar{w},\bar{\psi}_x,\bar{\psi}_y\} = L_1\{\bar{w},\bar{\psi}_x,\bar{\psi}_y\} - r\bar{\psi}_{x,tt} = 0, \tag{1}$$

$$\mathscr{S}_2\{\bar{w},\bar{\psi}_x,\bar{\psi}_y\} = L_2\{\bar{w},\bar{\psi}_x,\bar{\psi}_y\} - r\bar{\psi}_{y,tt} = 0, \tag{2}$$

$$\mathscr{S}_3\{\bar{w},\bar{\psi}_x,\bar{\psi}_y\} = L_3\{\bar{w},\bar{\psi}_x,\bar{\psi}_y\} + \bar{\bar{n}}(\bar{w}_{,xx} + \gamma\bar{w}_{,yy}) + \bar{p} - \bar{M}w_{,tt} = 0, \tag{3}$$

where the differential expressions are

$$L_1\{\bar{w},\bar{\psi}_x,\bar{\psi}_y\} = D_{11}\bar{\psi}_{x,xx} + (D_{12}+D_{66})\bar{\psi}_{y,xy} + D_{66}\bar{\psi}_{x,yy} - \kappa^2 A_{55}(\bar{w}_{,x}+\bar{\psi}_x), \tag{4}$$

$$L_2\{\bar{w},\bar{\psi}_x,\bar{\psi}_y\} = D_{22}\bar{\psi}_{y,yy} + (D_{12}+D_{66})\bar{\psi}_{x,xy} + D_{66}\bar{\psi}_{y,xx} - \kappa^2 A_{44}(\bar{w}_{,y}+\bar{\psi}_y), \tag{5}$$

$$L_3\{\bar{w},\bar{\psi}_x,\bar{\psi}_y\} = \kappa^2 A_{55}(\bar{w}_{,xx}+\bar{\psi}_{x,x}) + \kappa^2 A_{44}(\bar{w}_{,yy}+\bar{\psi}_{y,y}). \tag{6}$$

The parameters are determined by homogenization; perfect bond of the laminates is understood:

$$(A_{ij}, D_{ij}) = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} Q_{ij}^{(k)}(1, z^2)\,\mathrm{d}z, \tag{7}$$

$$(\bar{M}, r) = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \rho^{(k)}(1, z^2)\,\mathrm{d}z, \tag{8}$$

$$Q_{ij}^{(k)} = \begin{cases} (c_{ij} - c_{i3}c_{3j}/c_{33})^{(k)} \ldots i, j = 1, 2 \\ c_{ij}^{(k)} \qquad\qquad\quad \ldots i, j = 4, 5, 6 \end{cases} \quad k = 1, 2, \ldots, N, \tag{9}$$

$$\gamma = (A_{22}/A_{11})^{1/2}. \tag{10}$$

$\kappa^2 = 5/6$ is a shear factor, and $\rho^{(k)}$ stands for the mass density of the kth layer; $c_{ij}^{(k)}$ are the elements of the elasticity tensor of the linear elastic orthotropic lamina k, and $\bar{p}$ denotes the lateral loading.

Furthermore, since large deflections are to be considered by means of the approximation of Berger (1955), the in-plane forces are approximated by hydrostatic tensile forces $\bar{\bar{n}}$. The resulting equations are of the form of eqns (1)–(3); see Wu and Vinson (1970) for the detailed derivation. Following the arguments of Wu and Vinson (1969a), Berger's normal force is not explicitly effected by the influence of shear.

If the plate edges are prevented from in-plane motions, then the uniform in-plane force $\bar{n}$ is computed from (see, e.g., Sathyamoorthy (1978))

$$\bar{n}(t) = -\frac{A_{11}}{2\Omega}\int_{\Omega} \bar{w}(\bar{w}_{,xx}+\gamma\bar{w}_{,yy})\,\mathrm{d}\Omega, \tag{11}$$

with Ω being the area of the plate domain.

In case of higher forcing frequency the assumption of linear distribution of the in-plane displacements has to be replaced by approximations according to higher-order theories. Herrmann and Achenbach (1967) developed the "effective stiffness theory", for example, and further improvements are due to Sun and Whitney (1973), Reddy and Chao (1982), and Reddy (1984). An extensive review of shear deformation theories for multilayered composite plates is presented by Noor and Burton (1989).

3. SPECTRAL FORMULATION OF NONLINEAR VIBRATIONS

Large deflections of plates which are loaded by a time-harmonic lateral force, $\bar{p}(x, y; t) = p(x, y)\sin\omega t$, are investigated. Adopting the method of Kantorovich (compare Szemplinska-Stupnicka (1983)) to the nonlinear plate problem we assume that the steady-state response is represented by

$$\bar{w}^*(x, y; t) = w(x, y)\sin\omega t, \tag{12}$$

$$\bar{\psi}_i^*(x, y; t) = \psi_i(x, y)\sin\omega t, \quad i = x, y, \tag{13}$$

where a "best choice" of functions $w(x, \bar{y})$, $\psi_i(x, y)$ has to be determined. Therefore, this time-harmonic approximation enters the differential operators of eqns (1)–(3). Finally the "error" is forced to be zero in the average sense of the Galerkin procedure,

$$\frac{1}{\omega}\int_0^{2\pi} \mathscr{D}_i\{\bar{w}^*, \bar{\psi}_x^*, \bar{\psi}_y^*\}\sin\omega t\,\mathrm{d}(\omega t) = 0, \quad i = 1, 2, 3, \tag{14}$$

and as a result the equations of motion in the frequency domain are obtained:

$$L_1(w, \psi_x, \psi_y\} + r\omega^2\psi_x = 0, \tag{15}$$

$$L_2\{w, \psi_x, \psi_y\} + r\omega^2\psi_y = 0, \tag{16}$$

$$L_3\{w, \psi_x, \psi_y\} + \tfrac{3}{4}n(w_{,xx}+\gamma w_{,yy}) + p + \bar{M}\omega^2 w = 0, \tag{17}$$

where n is defined by eqn (11) when $\bar{w}$ is replaced by the time-reduced counterpart w.

In particular, for plates which are composed of multiple isotropic layers, eqns (15)–(17) can be reduced to a set of Herrmann–Armenakas-type equations for a homogeneous Mindlin-plate with effective stiffnesses and hydrostatic in-plane force; see Heuer (1991). In that case, after eliminating the cross-sectional rotations, a single fourth-order differential equation for $w(x, y)$ is obtained, which corresponds to the equation of a fictitious Lagrange–Kirchhoff plate with bending stiffness $\bar{K}$, uniform inplane force $\bar{n}$, and inertia terms $\bar{\mu}$:

$$\bar{K}\Delta\Delta w - \bar{n}\Delta w - \bar{\mu}w = \bar{q} - \bar{K}(1+\nu)\Delta\bar{\kappa}. \tag{18}$$

The coefficients and the forcing terms are

$$\bar{K} = D_{11}[1+\tfrac{3}{4}ns], \tag{19}$$

$$\bar{n} = \tfrac{3}{4}n - \omega^2[D_{11}\bar{M}s + r + \tfrac{3}{4}nrs], \tag{20}$$

$$\bar{\mu} = \bar{M}\omega^2(1-sr\omega^2), \tag{21}$$

$$\bar{q} = p(1-sr\omega^2), \tag{22}$$

$$\bar{\kappa} = ps/\{[1+\tfrac{3}{4}ns](1+\nu)\}, \tag{23}$$

$$n(\omega) = -\frac{A_{11}}{2\Omega}\int_\Omega w\Delta w \, d\Omega, \tag{24}$$

$$s = 1/\kappa^2 A_{44}, \quad \nu = D_{12}/D_{11}. \tag{25}$$

For vanishing imposed loading, eqn (18) provides the "nonlinear normal modes" of free vibrations. The boundary conditions of shear deformable plates with hard hinged supports can be modelled in the form (see Mindlin (1951)):

$$C: w = 0, \quad \psi_s = 0, \quad m_n = D_{11}(\psi_{n,n}+\nu\psi_{s,s}) = 0, \tag{26}$$

with the bending moment m and a local (n, s)-coordinate system at boundary C with normal (n). Considering only polygonal contours C the third condition of eqn (26) simplifies to that at a straight edge segment, $\psi_{s,s} = \psi_{n,n} = 0$. Inserting these relations into eqn (17) finally leads to two boundary conditions in w:

$$C: w = 0, \quad \Delta w = -\bar{\kappa}(1+\nu). \tag{27}$$

The complete boundary value problem determined by eqns (18) and (27) is efficiently solved by a membrane analogy. The latter has been worked out for linear free and forced vibrations of homogeneous Mindlin-plates by Irschik (1985) and by Irschik *et al.* (1988, 1989). In the course of this method of analysis the differential equation (18) of the total deflection is decomposed into a set of two second-order problems, each corresponding to the boundary value problem of the deflection of a prestressed linear elastic membrane:

$$w = w_1 + w_2, \tag{28}$$

$$\Delta w_j + \alpha_j w_j = -\alpha_j \theta_j p, \tag{29}$$

where

$$\alpha_j = -[\bar{n}+(-1)^j(\bar{n}^2+4\bar{K}\bar{\mu})^{1/2}]/2\bar{K}, \quad j = 1, 2. \tag{30}$$

Inserting eqns (28), (29) into eqn (18) and comparing the coefficients of p and Δp, respectively, renders two equations for the yet unknown coefficients θ_j:

$$\alpha_1^2\theta_1 + \alpha_2^2\theta_2 + \bar{n}[\alpha_1\theta_1 + \alpha_2\theta_2]/\bar{K} = (1-sr\omega^2)/\bar{K}, \tag{31}$$

$$\alpha_1\theta_1 + \alpha_2\theta_2 = s/[1+\tfrac{3}{4}ns]. \tag{32}$$

Their solution gives at once

$$\alpha_j\theta_j = +(-1)^j\{(1-sr\omega^2)/\bar{K}+\alpha_j s/[1+\tfrac{3}{4}ns]\}/(\alpha_1-\alpha_2). \tag{33}$$

Subsequently, eqns (28) and (29) are inserted into the boundary conditions (27), and considering eqn (32) leads to the homogeneous boundary conditions of the associated membrane problems:

$$C: w_j = 0, \quad j = 1,2. \tag{34}$$

Berger's integral equation (24), which bears the nonlinearity, is reformulated by means of eqns (28), (29) to become:

$$n(\omega) = -\frac{A_{11}}{2\Omega}\int_{\Omega}(w_1+w_2)[(-\alpha_1\theta_1 p-\alpha_1 w_1)+(-\alpha_2\theta_2 p-\alpha_2 w_2)]\,d\Omega. \tag{35}$$

Hence, the underlying boundary value problem is, with respect to deflections w, governed by the two second-order Helmholtz-type equations (29) with Dirichlet's conditions (34).

4 NUMERICAL RESULTS AND COMPUTATIONAL ASPECTS

For a fixed excitation frequency ω, eqn (35) is a nonlinear equation for the in-plane force n, since the formal solution of the linear boundary value problems (28), (29) provide w_1, w_2 as functions of n. Thus, the geometrically nonlinear problem is reduced to a single nonlinear equation, which, dependent on the range of frequency, has one, three or more solutions. These roots are determined by means of numerical standard procedures.

The method is demonstrated for a plate strip, where in case of $p = p_0 = \text{const.}$ (compare Chwalla (1962)):

$$w_j = p_0\frac{\alpha_j\theta_j}{\alpha_j}(\{1-\tfrac{1}{2}[\exp(i\sqrt{\alpha_j}l)+\exp(-i\sqrt{\alpha_j}l)]\}[\exp(i\sqrt{\alpha_j}x)-\exp(-i\sqrt{\alpha_j}x)]/[\exp(i\sqrt{\alpha_j}l)$$
$$-\exp(-i\sqrt{\alpha_j}l)]+\tfrac{1}{2}[\exp(i\sqrt{\alpha_j}x)+\exp(-i\sqrt{\alpha_j}x)]-1), \quad i=\sqrt{-1}, \quad j=1,2, \tag{36}$$

and the coefficients $\alpha_j(n)$ and $\alpha_j\theta_j(n)$ are defined according to eqns (30) and (33), respectively. After analytical determination of Berger's integral, eqn (35) is solved iteratively by the "regula falsi" algorithmus. Its solutions $n_k(\omega)$ are inserted into eqn (36) and the nonlinear frequency response function (FRF) is finally obtained by superposition of w_1 and w_2. As an illustrative example, the FRFs of a plate strip (length l, total plate thickness h, and stiffnesses D_{11} and D_{12}) are evaluated. Passing over to a non-dimensional formulation, the input parameters are specified through $\bar{h} = h/l = 0.15$, $\nu = D_{12}/D_{11} = 0.3$, and a non-dimensional forcing frequency is defined by

$$\lambda = \omega lh\sqrt{\bar{M}/6(1-\nu)D_{11}}. \tag{37}$$

Figure 1 shows the nonlinear FRFs $\bar{w} = w(x = l/2, \lambda)/h$ within the frequency range,

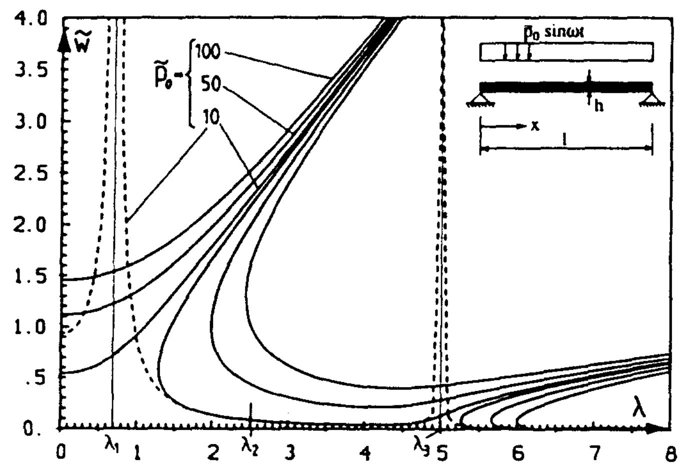

Fig. 1. Linear (----) and nonlinear (——) non-dimensional undamped frequency response function $\bar{w} = w(x = l/2, \lambda)/h$ as a function of $\lambda = \omega lh\sqrt{\bar{M}/6(1-\nu)D_{11}}$; $\bar{p}_0 = p_0 l^3/D_{11} = 10, 50, 100$, and $\bar{h} = h/l = 0.15$, $\nu = D_{12}/D_{11} = 0.3$.

$0 < \lambda \leqslant 8$, for various load factors $\tilde{p}_0 = p_0 l^3/D_{11} = 10, 50, 100$, compared to the linear FRF for $\tilde{p}_0 = 10$. Note that due to the uniform load distribution only the symmetric vibration modes (1 and 3 in the above given frequency band) are excited.

5. CONCLUDING REMARKS

The problem of forced nonlinear vibrations of layered plates is solved. The influence of shear and rotatory inertia is taken into account. Using Berger's approximation of v. Karman's nonlinear plate equations along with a Kantorovich–Galerkin procedure renders the frequency domain formulation of this problem. For laminates composed of multiple isotropic layers an efficient membrane analogy, which has been established for homogeneous plates, is adopted. As numerical example the nonlinear frequency response functions of a composite plate strip under harmonic uniform force-excitation are compared to the linear solution.

REFERENCES

Bennett, J. A. (1971). Nonlinear vibration of simply supported angle ply laminated plates. *AIAA Jl* **10**, 1997–2003.

Berger, H. M. (1955). A new approach to the analysis of large deflections of plates. *J. Appl. Mech.* **22**, 465–472.

Chwalla, E. (1962). Second-order theory of structures. In *Handbook of Engineering Mechanics* (Edited by Flügge, W.), pp. 30.1–30.23. McGraw-Hill, New York.

Herrmann, G. (1955). Influence of large amplitudes on flexural motions of elastic plates. *NACA TN* **3578**.

Herrmann, G. and Achenbach, J. D. (1967). On dynamic theories of fiber-reinforced composites. In *Proc. AIA Aerospace Am. Soc. Mech. Engrs 8th Structures, Structural Dynamics and Materials Conf.*, 1967, Palm Springs, California, pp. 112–118.

Herrmann, G. and Armenakas, A. E. (1960). Vibration and stability of plates under initial stress. *Proc. ASCE, J. Engng Mech. Div.* **86**, 65–94.

Heuer, R. (In press). A correspondence between multilayered shear deformable plates and homogeneous Mindlin-plates.

Heuer, R., Irschik, H. and Ziegler, F. (1990). A BEM-formulation of nonlinear plate vibrations. In *Proc. IUTAM/ICTAM-Symp. on Discretization Methods in Structural Mechanics*, Vienna 1989 (Edited by Kuhn, G., Mang, H.), pp. 341–351. Springer, Berlin.

Irschik, H. (1985). Membrane-type eigenmotions of Mindlin plates. *Acta Mech.* **55**, 1–20.

Irschik, H. (1990). Influence of large amplitudes on free flexural vibrations of polygonal shear-deformable plates—a unifying dimensionless formulation. *Int. J. Solids Structures* **26**, 675–681.

Irschik, H., Heuer, R. and Ziegler, F. (1988). Free and forced vibrations of polygonal Mindlin-plates by an advanced BEM. In *Proc. IUTAM Symp. on Advanced Boundary Element Methods*, San Antonio 1987 (Edited by Cruse, T. A.), pp. 179–188. Springer, Berlin.

Irschik, H., Heuer, R. and Ziegler, F. (1989). Dynamic analysis of polygonal Mindlin plates on two-parameter foundations using classical plate theory and an advanced BEM. *Comp. Mech.* **4**, 293–300.

Kollbrunner, C. F. and Herrmann, G. (1956). Einfluß des Schubes auf die Stabilität der Platten im elastischen Bereich. *Mitteilungen der T.K.V.S.B.* **14**. V.S.B., Zürich.

Mindlin, R. D. (1951). Influence of rotatory inertia and shear on flexural motions of isotropic, elastic plates. *J. Appl. Mech.* **18**, 31–38.

Noor, A. K. and Burton, W. S. (1989). Assessment of shear deformation theories for multilayered composite plates. *Appl. Mech. Rev.* **42**, 1–13.

Nowinski, J. L. and Ohnabe, H. (1972). On certain inconsistencies in Berger equations for large deflections of elastic plates. *Int. J. Mech. Sci.* **14**, 165–170.

Reddy, J. N. (1984). A simple higher-order theory for laminated composite plates. *J. Appl. Mech.* **51**, 745–752.

Reddy, J. N. and Chao, W. C. (1982). Nonlinear oscillations of laminated, anisotropic, rectangular plates. *J. Appl. Mech.* **49**, 396–402.

Sathyamoorthy, M. (1978). Vibration of plates considering shear and rotatory inertia. *AIAA Jl* **16**, 285–286.

Schmidt, R. (1974). On Berger's method in the nonlinear theory of plates. *J. Appl. Mech.* **41**, 521–523.

Sun, C. T. and Whitney, J. M. (1973). Theories for the dynamic response of laminated plates. *AIAA Jl* **11**, 178–183.

Szemplinska-Stupnicka, W. (1983). "Non-linear normal modes" and the generalized Ritz method in the problems of vibrations of non-linear elastic continuous systems. *Int. J. Non-Linear Mech.* **18**, 149–165.

Wah, T. (1963). Large amplitude flexural vibration of rectangular plates. *Int. J. Mech. Sci.* **5**, 425–438.

Whitney, J. M. and Pagano, N. J. (1970). Shear deformation in heterogeneous anisotropic plates. *J. Appl. Mech.* **37**, 1031–1036.

Whitney, J. M. and Leissa, A. W. (1969). Analysis of heterogeneous anisotropic plates. *J. Appl. Mech.* **36**, 261–266.

Wu, C.-I. and Vinson, J. R. (1969a). Influences of large amplitudes, transverse shear deformation, and rotatory inertia on lateral vibrations of transversely isotropic plates. *J. Appl. Mech.* **36**, 254–260.

Wu, C.-I. and Vinson, J. R. (1969b). On the nonlinear oscillations of plates composed of composite materials. *J. Composite Mater.* **3**, 548–561.

Wu, C.-I. and Vinson, J. R. (1970). Nonlinear oscillations of laminated specially orthotropic plates with clamped and simply supported edges. *J. Acoust. Soc. Amer.* **49**, 1561–1567.